10 Rotations of the Plane

You've learned about matrix multiplication and about complex numbers. You may have guessed at the relationship between them, particularly now that we've spent some time seeing how 2×2 matrices, certain 3×3 matrices, and complex numbers all relate to the geometry of the plane. We will now make that connection explicit and relate it to some ideas that you worked with last year, such as rotation of coordinates.

Recall that a matrix M acts on a column vector v by the multiplication Mv, not vM; a complex number z acts on the point (x, y) by multiplication $z(x + yi)$ or $(x + yi)z$, since complex multiplication is commutative. We'll call v or (x, y) the preimage and Mv or $z(x + yi)$ the image.

Some of the following problems are really trivial, so don't be alarmed if your answer takes only a few seconds. Some of them are fairly difficult and will take a bit of thought. Some of them are fairly tedious and will take some lengthy algebra, but not much thought.

1. (a) Which matrix changes nothing, so that the image is the same as the preimage?
   (b) Which complex number changes nothing?
2. (a) Which matrix doubles the length of every vector but leaves angles unchanged?
   (b) Which complex number corresponds to the same transformation?
3. Based on your answers to the previous problems, which matrix corresponds to the real number r? Let's call this $\mathrm{M}(r)$ for short.
4. Explain why $\mathrm{M}(u) + \mathrm{M}(v) = \mathrm{M}(u + v)$.
5. Under a $90°$ counterclockwise rotation, what is the image of (a) $(1, 0)$ and (b) $(0, 1)$?
6. (a) Which matrix corresponds to a $90°$ rotation?
   (b) Which complex number corresponds to the same rotation?
7. Based on your answers to Problems 1–6, what matrix corresponds to the complex number $x + yi$? Let's extend our function M and call this $\mathrm{M}(x + yi)$ for short.
8. Check that $\mathrm{M}(a+bi) + \mathrm{M}(c+di) = \mathrm{M}((a+bi) + (c+di))$. That is, prove that M has the same addition rules as complex numbers.
9. Check that $\mathrm{M}(a+bi)\,\mathrm{M}(c+di) = \mathrm{M}((a+bi)(c+di))$. That is, prove that M has the same multiplication rules as complex numbers.
10. Recall that multiplying by $\operatorname{cis}\theta$ rotates a complex number by θ radians.
    (a) Find $\mathrm{M}(\operatorname{cis}\theta)$.
    (b) To prove that this matrix really does rotate by θ:
        i. Check that the image and preimage have the same length.
        ii. Check that the angle of the image with the x axis is θ more than the preimage.
11. (a) Find $\mathrm{M}(r \operatorname{cis}\theta)$.
    (b) To prove that this matrix really does rotate by θ and stretch by r:
        i. Check that the length of the image is r times the length of the preimage.
        ii. Check that the angle of the image with the x axis is θ more than the preimage. (Hint: you may want to use the previous problem, or the tangent addition formulas.)

We've seen that there is a matrix for every complex number. These matrices have the same addition and multiplication rules as complex numbers. Furthermore, these matrices transform the plane in the same way as complex multiplication: a stretch by a factor of r and a rotation by θ. There are many matrices, however, that don't correspond to complex numbers.

12. (a) What matrix reflects over the x axis, taking $(x, y) \to (x, -y)$?
    (b) What is the complex number *operation* equivalent to this transformation?
    (c) Is there a complex number multiplication equivalent to this transformation? Justify your answer.

13. (a) What matrix reflects through the origin, taking $(x, y) \to (-x, -y)$?
    (b) What is the complex number operation equivalent to this transformation?
    (c) Is there a complex number multiplication equivalent to this transformation? Justify your answer.
14. (a) Which of the 16 matrices on page 26, for Problem 10, have corresponding complex numbers?
    (b) How can you tell algebraically?
    (c) How can you tell geometrically?
15. Make multiplication tables with the set of matrices which correspond to the elements of the rotation group for the square (a 4×4 table) and the equilateral triangle (a 3×3 table).
16. (a) Write a matrix for a rotation of θ around the origin followed by a translation by (a, b).
    (b) Write a matrix for a translation by (a, b) followed by a rotation of θ around the origin.

Now that we know how to rotate with matrices and with complex numbers, we can revisit the topic of rotation of axes that you studied toward the end of last year.

17. Use matrix multiplication to find the image (x', y') of a point (x, y) rotated by θ.
18. (a) Given the parabola $x = t, y = t^2$, use matrix multiplication to rotate it by $45°$ counterclockwise.
    (b) Graph the new parametric equations on your calculator.